A PARTICULAR SOLUTION OF THE PROBLEM OF THE MOTION OF A GYROSCOPE ON GIMBALS

(CHASTNOE RESHENIE ZADACHI O DVIZHENII GIROSKOPA V KARDANOVOM PODVESE)

PMM Vol.23, No.5, 1959, pp. 958-960

A.A. BOGOIAVLENSKII
(Moscow)

(Received 28 April 1959)

The motion of a symmetric heavy gyroscope on gimbals, for the case in which the axis of the outer ring is vertical, was investigated by Chetaev [1], Skimel' [2], Magnus [3] and Rumiantsev [4], who considered the stability of motion for certain particular solutions.

For the case of a gyroscope on gimbals, the axis of the outer ring being horizontal, questions of stability of certain particular motions were considered [5]. In adding for a certain moment of external forces with respect to the axis of spin of the gyroscope, a first integral was obtained [5].

Below, a particular solution of the problem of motion of a heavy gyroscope on gimbals is investigated, for the case in which the axis of rotation of the outer ring is horizontal.

1. Let us consider a symmetric gyroscope on gimbals in which the axis of rotation of the outer ring of the Cardan suspension is horizontal and the center of gravity of the gyroscope and the inner ring is located on the axis of symmetry of the gyroscope.

We introduce two rectangular systems of coordinate axes, with origin at the fixed point O of the gyroscope (see Fig. 1).

The fixed system of coordinates $Ox_1y_1z_1$ is always connected with the axis of rotation of the outer gimbal ring. The axis x_1 is directed vertically upward, the axis z_1 along the axis of rotation of the outer ring. The moving system of coordinates $Oxyz$ is always connected with the inner ring. The x-axis is directed along the axis of rotation of the inner ring, the z-axis along the axis of symmetry of the gyroscope.

The motion of the gyroscope with respect to the system of axes $x_1y_1z_1$ is determined by the following angles: ψ is the angle of rotation of the outer gimbal ring, θ is the angle of rotation of the inner ring, ϕ is the angle of rotation of the gyroscope within the inner ring (the angle of spin of the gyroscope with respect to the xyz system of coordinates).

The projections of the instantaneous velocity of rotation of the inner ring on the axes of coordinates $Oxyz$ are equal to

$$p^\circ = \theta', \qquad q^\circ = \psi' \sin\theta, \qquad r^\circ = \psi' \cos\theta$$

The projections of the instantaneous angular velocity of the gyroscope on the same axes are expressed by relations

$$p = \theta', \qquad q = \psi' \sin\theta, \qquad r = \varphi' + \psi' \cos\theta$$

Let the axes x, y, z be the principal axes of the ellipsoid of inertia of the inner ring with respect to the fixed point O.

Let us designate by ζ the distance from the origin O to the center of gravity of the gyroscope and the inner ring, let I be the moment of inertia of the inner gyroscope ring with respect to the z_1 axis, let A^0, A^0, C^0 be the principal moments of inertia of the inner ring with respect to the axes x, y, z, let A, $B = A$, C be the moments of inertia of the gyroscope with respect to the same axes.

Let us assume that the ellipsoid of inertia of the gyroscope with respect to the point O is an ellipsoid of rotation with respect to the axis Oz. The expression of twice the kinetic energy of the outer gimbal ring, the inner ring and the gyroscope are respectively equal to

$$I\psi'^2, \qquad A^\circ p^{\circ 2} + B^\circ q^{\circ 2} + C^\circ r^{\circ 2}, \qquad Ap^2 + Bq^2 + Cr^2$$

The total kinetic energy of the system is

$$2T = (A + A^\circ)\,\theta'^2 + [I + C^\circ + (A + B^\circ - C^\circ)\sin^2\theta]\,\psi'^2 + C\,(\varphi' + \psi'\cos\theta)^2$$

We shall assume that there is no friction in the bearings and that the applied forces acting on the system reduce to the forces of gravity. Let m be the mass of the gyroscope and the inner ring. Then the force function is of the form

$$U = -\,mg\zeta \sin\theta \sin\psi$$

2. Let the mass distribution of the considered system be such that

$$A + A^\circ = I + C^\circ, \qquad A + B^\circ = C^\circ$$

We introduce the notation $I_1 = A + A^0$. Then the equation of motion of the system may be written down in the form of Lagrange's equations for the independent holonomic variables θ, ψ, ϕ:

$$\begin{gathered} I_1\theta'' + C\,(\varphi' + \psi'\cos\theta)\sin\theta\,\psi' = -\,mg\zeta\cos\theta\sin\psi \\ \frac{d}{dt}\{I_1\psi' + C\,(\varphi' + \psi'\cos\theta)\cos\theta\} = -\,mg\zeta\sin\theta\cos\psi \\ \frac{d}{dt}\,C\,(\varphi' + \psi'\cos\theta) = 0 \end{gathered} \tag{1}$$

The actual displacements of the system are found among the possible ones, and the forces admit a force function. From this follows the expression of the integral of kinetic energy

$$I_1(\theta'^2 + \psi'^2) + C(\varphi' + \psi'\cos\theta)^2 = -2mg\zeta\sin\theta\sin\psi + 2h \tag{2}$$

The cyclic coordinate ϕ corresponds to the first integral

$$\varphi' + \psi'\cos\theta = r_0 \tag{3}$$

Here h, r_0 are the constants of the indicated first integrals.

For a particular value of the constant of the integral (3), namely when it is equal to zero, i.e. in case

$$\varphi' + \psi'\cos\theta = 0 \tag{4}$$

there exists one more first integral

$$I_1\theta'\psi' = mg\zeta\cos\theta\cos\psi + l \qquad (l = \text{const}) \tag{5}$$

3. Let us introduce the variables α, β, which are related to the angles θ, ψ as

$$\alpha = \theta + \psi, \qquad \beta = \theta - \psi \tag{6}$$

This substitution leads to the separation of variables. The equations of motion of the system considered, in the presence of the particular integral (4), may be then written in the form

$$I_1\alpha'' = -mg\zeta\sin\alpha, \quad I_1\beta'' = mg\zeta\sin\beta \tag{7}$$

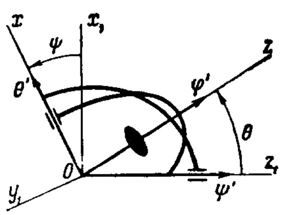

Fig. 1.

These differential equations admit the first integrals

$$\begin{aligned} I_1(\alpha'^2 + \beta'^2) &= 2mg\zeta(\cos\alpha - \cos\beta) + 4h \\ I_1(\alpha'^2 - \beta'^2) &= 2mg\zeta(\cos\alpha + \cos\beta) + 4l \end{aligned} \tag{8}$$

which may be obtained also from integral (2) and (5).

Let us introduce the following designation for the constants:

$$a = \frac{h+l}{3I_1}, \qquad b = \frac{h-l}{3I_1}, \qquad c = \frac{mg\zeta}{I_1}$$

From integrals (8) it follows

$$\alpha'^2 = 2(c\cos\alpha + 3a), \qquad \beta'^2 = -2(c\cos\beta - 3b) \tag{9}$$

We introduce new variables u, v by means of formulas

$$-2u = c\cos\alpha + a, \qquad 2v = c\cos\beta - b \tag{10}$$

In these variables the equations (9) are written as

$$\left(\frac{du}{dt}\right)^2 = 4u^3 - (3a^2 + c^2)\,u - a\,(a^2 - c^2), \qquad \left(\frac{dv}{dt}\right)^2 = 4v^3 - (3b^2 + c^2)\,v - b\,(b^2 - c^2) \tag{11}$$

We construct the Weierstrass functions $\gamma_1(\tau)$ with the invariants

$$g_2' = 3a^2 + c^2, \qquad g_3' = a\,(a^2 - c^2)$$

and $\gamma_2(\tau)$ with the invariants

$$g_2'' = 3b^2 + c^2, \qquad g_3'' = b\,(b^3 - c^2)$$

These functions satisfy the equation

$$\wp'^2(\tau) = 4\wp^3(\tau) - g_2\wp(\tau) - g_3$$

If we set

$$\begin{aligned} u &= \wp_1(\tau), & -2\wp_1(\tau) &= c\cos\alpha + a \\ v &= \wp_2(\tau), & 2\wp_2(\tau) &= c\cos\beta - b \end{aligned} \tag{12}$$

where τ is a function of time t, then the equations (11) yield

$$\wp_1'^2(\tau)\left(\frac{d\tau}{dt}\right)^2 = 4\wp_1^3(\tau) - g_2'\wp_1(\tau) - g_3', \qquad \wp_2'^2(\tau)\left(\frac{d\tau}{dt}\right)^2 = 4\wp_2^3(\tau) - g_2''\wp_2(\tau) - g_3''$$

Therefore

$$\left(\frac{d\tau}{dt}\right)^2 = 1, \qquad \frac{d\tau}{dt} = \pm 1$$

Taking the positive sign we obtain

$$\tau = t + t_0 \qquad (t_0 = \mathrm{const})$$

In accordance with formulas (6) and (12) we find

$$\begin{aligned} \theta &= \frac{1}{2}\left\{\arccos\left[-\frac{2I_1}{mg\zeta}\wp_1(t+t_0) - \frac{h+l}{3mg\zeta}\right] + \arccos\left[\frac{2I_1}{mg\zeta}\wp_2(t+t_0) + \frac{h-l}{3mg\zeta}\right]\right\} + \lambda \\ \psi &= \frac{1}{2}\left\{\arccos\left[-\frac{2I_1}{mg\zeta}\wp_1(t+t_0) - \frac{h+l}{3mg\zeta}\right] - \arccos\left[\frac{2I_1}{mg\zeta}\wp_2(t+t_0) + \frac{h-l}{3mg\zeta}\right]\right\} + \mu \end{aligned} \tag{13}$$

where λ, μ are constants, whose values are determined by the solution found, and by the domain of variation of θ, ψ.

The angle ϕ is calculated by the quadrature from the relation (4).

4. The variables u, v are determined by relationships (10) and satisfy equations (11). Let us designate the right-hand sides of these equations, which are third degree polynominals by $f(\xi)$

$$f(\xi) = 4\xi^3 - (3\varkappa^2 + c^2)\,\xi - \varkappa\,(\varkappa^2 - c^2) \tag{14}$$

where $\kappa = a$ for $\xi = u$ and $\kappa = b$ for $\xi = v$.

The polynomial (14) has three real roots

$$e_1 = -\frac{c+\varkappa}{2}, \qquad e_2 = \frac{c-\varkappa}{2}, \qquad e_3 = \varkappa$$

Let us consider $c > 0$. From relations (10) it follows that the variables u, v, determined by means of ξ, vary in the range

$$e_1 \leqslant \xi \leqslant e_2$$

Actual mechanical motions will take place if $e_3 > e_1$.

This condition yields the inequality for the variable u

$$h + l + mg\zeta > 0$$

and for the variable v - the inequality

$$h - l + mg\zeta > 0$$

In like manner one can show that for $c < 0$ actual mechanical motions will take place if the inequalities are satisfied

$$h + l - mg\zeta > 0, \qquad h - l - mg\zeta > 0$$

A study of the stability of particular solutions of system (1) does not yield anything essentially new as compared to the conditions found earlier [2,4].

BIBLIOGRAPHY

1. Chetaev, N.G., O gyroskope v kardanovom podvese (On the Gyroscope on Gimbals). *PMM* Vol. 27, No. 3, pp. 379-381, 1958.

2. Skimmel', B.N., Nekotorye zadachi dvizheniia i ustoichivosti tiazhelogo giroscopa (Certain problems of motion and stability of a heavy gyroscope). *Trudy KAI* 38, pp. 103-129, 1958.

3. Magnus, K., Ob ustoichivosti dvizheniia tiazhelogo symmetrichnogo giroskopa v kardanovom podvese (On the stability of a heavy symmetric gyroscope on gimbals). *PMM* Vol. 22, No. 2, pp. 173-178, 1958.

4. 5. Rumiantsev, V.V., Ob ustoichivosti dvizheniia gyroskopa v kardanovom podvese (On the stability of motion of a gyroscope on gimbals). *PMM* Vol. 22, No. 3, pp. 374-378, 1958; *PMM* Vol. 22, No. 4, pp. 499-503, 1958.

Translated by G.H.